

T H E
E L E M E N T S

O F

A NEW METHOD OF REASONING

I N

G E O M E T R Y:

APPLIED TO THE RECTIFICATION OF THE CIRCLE.

By T H O M A S T A Y L O R.

—*Nec altum sapiamus, nec ultra sobrium,
sed veritatem in charitate colamus.*

BACON.

L O N D O N:

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E R R A T U M,

Page 7. Line 5. *Instead of "the χ Arch," read "the Arch,"*



A D V E R T I S E M E N T.

THE Author of the following small Tract is not ashamed to confess, that it has been the employment of his leisure hours for a considerable time. If he has failed in the execution, he can, however, safely affirm he has not been wanting in the most earnest endeavours towards the completion of his purpose. He considered that the object of his search and enquiry, although arduous, was at the same time glorious, and that the Discovery of Truth is always a sufficient recompence for the difficulty attending its Investigation. As he cannot, by the most impartial scrutiny, detect any false Reasoning in his Demonstrations, he flatters himself they will not be found altogether destitute of support, nor wholly unworthy the
appro-

approbation of the Public. However, sensible of his own weakness, he would not too confidently presume on success, since in this case to DESIRE is not sufficient TO OBTAIN.

In short, animated by a sincere Love of TRUTH, he flatters himself the integrity of his Intentions will in some measure atone for his want of greater genius and abilities.

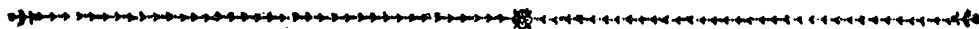
Indeed, as an elegant Writer observes, since Mediocrity is now become a Protection, he has probably obtained THAT PROTECTION in spite of himself. He only adds (with the same Author) and would wish the Reader to remember, that while Error sinks into the abyss of forgetfulness, TRUTH alone swims over the vast extent of ages.

THE



T H E

E L E M E N T S, &c.



L E M M A.

IF there be any three Quantities, X , Y , Z , the former of which X , is greater than either of the other two, Y , Z , then it cannot be said Z is greater than Y , by the excess of X above Y ; but if it is at all greater, it must be by some Quantity less than the excess of X above Y .

For if it is denied, Z shall be equal to X , which is absurd.

PROPOSITION I. *Fig. I.*

If from the same Centre s , any two Arches tR , $a\chi$, are described as in the Figure, and with the Radius sa , of the greatest $a\chi$,

B

the

the Arch $t z$ is drawn touching the Arch $R t$, in the Point t , then the Arch $R t$, shall not be greater than the Arch $t z$.

From the Point z , draw the Line $z p$ parallel to the Line $s a$; then, as is well known, the Arch $p a$ shall be equal to the Arch $z t$. Suppose the Arch $z p$ to be divided by a continual Bisection into an infinite number of equal Parts, at the points $o, o, o, \&c.$ and let $o \chi$ represent one of such parts: From o draw the right line $o o$, parallel to $a s$, and with a radius equal to $a s$ describe the arch λo , and from the centre s , the arch λs . Now let (M) represent the infinitely small quantity χo , then if the arch λs exceed the Arch λo , by the quantity M, it shall be equal (from the Lemma) to the Arch $a \chi$, which is absurd. The same consequences will evidently follow, if a Quantity less than χo is assigned, and so the Arch λs is not greater than the Arch λo .

Again, Suppose the Arch λo , to exceed the Arch $m \lambda$ by the like Quantity (M), then if the Arch $m \lambda$ exceed $m \lambda$, by (M), it shall be equal to the Arch λo . But the Arch λs , is not greater than λo from the Demonstration, and so in this case is not greater than the Arch $m \lambda$, which is evidently absurd. In like manner proceeding infinitely, we shall at last prove, that the Arch $t R$ is not greater than the Arch $t z$. Q. E. D.

Fig. 1.

Fig. 1. It must be carefully observed that this method of Reasoning will not take place, if the Angle $a s \pi$ is assumed greater than a right Angle, (suppose the Angle $a s G$). For since it is always required to the Demonstration that an infinitely small Part of the Arch $a G$ must be taken, and a Line drawn parallel to $s a$. If from the Point F , which terminates the Quadrant $a F$, an infinitely small Arch $F o$ is taken, and a Line drawn parallel to $a s$, the Line so drawn shall be a Tangent to the Arch FG , and so fall without the Curve.—In like manner, any Line drawn from a Point below F , parallel to $a s$, shall fall without the Curve towards G ; and therefore in this case the preceding Demonstration cannot take place.

COROLLARY I.

Fig. 1. No. II. It may not, perhaps, be unnecessary to observe, that after the same manner as in this Proposition, it is easy to prove that in the Quadrant $a K$, if any right line ϕo is drawn parallel to χa , then any other right Line ϕp shall not be greater than ϕo .

COR. II.

Fig. 1. No. III. If Concentric and Tangent Arches are described as in the Figure terminated by the Quadrantal Arch Πs ,
C
then

then from this Proposition, (if we suppose the concentric Arches from s towards π continually increase) any Concentric Arch χy shall not be greater than it's Tangent Arch σs .

COR. III.

Hence, (the same Things remaining as in COR. II.) if $m\theta$ is described exceeding $\Delta\sigma$ by (M), and if $m\lambda = m\theta$, then all the concentric Arches above $\Delta\sigma$ shall equal their correspondent Tangent Arches, described with the Radius $s\sigma$ as per Scheme.

For, suppose χs to exceed $m\theta$ by (M), then if $m\lambda = m\theta$, the Arch χy shall be greater than $m\theta$, and not greater than χs (Pr. I.) Therefore χy cannot be less than χs and so must be equal to it.

Q. E. D.

It were easy to infer a Variety of other Conclusions from this Proposition, but as they will naturally present themselves to the Reader's observation, without much enquiry, they are omitted in this place.

PROPO-



PROPOSITION II. *Fig. 2.*

I From the Center c , with the Radius $c x$, the Arch $x \lambda$ is described, and from the Centers a, c, s , the respective Tangent Arches, $x \mu, x \nu, x \rho$, &c. Then the Arch $x \lambda$ shall be less than the Arch $x \nu$.

For from PROP. I. $x \lambda$ is not greater than $x \nu$, and from the same PROP. $x \mu$ is not greater than $x \nu$; $x \mu$ is, therefore less than $x \nu$. But $x \lambda$ is likewise not greater than $x \mu$; it is therefore either equal to, or less than $x \mu$, in each of which cases it shall be less than $x \nu$. Therefore I conclude, universally, that any concentric Arch $x \lambda$ is less than its Tangent Arch $x \mu$, terminated by the same right Line $x \mu$. Q. E. D.

COR. I.

Hence we have an easy method of approximating continually nearer and nearer to an Equality between two Arches, $x \rho$, and $x \lambda$. Draw the right Line $o \mu x$, intersecting the Arch $x \rho$ in s ; then because $x \mu$ is less than $x \nu$, and $x \lambda$ less than $x \mu$, $x \lambda$ shall be less than $x s$; through the Point o , Draw the right Line $t a w$, then $x o$ shall

shall be less than $\kappa\pi$: But $\kappa\mu$ has been proved to be less than $\kappa\sigma$, and $\kappa\lambda$ less than $\kappa\mu$; therefore $\kappa\lambda$ shall be less than $\kappa\pi$, and so on infinitely.

COR. II.

Hence likewise, we may easily approximate to an Equality between a right Line, and any circular Arch. For it is well known that any Arch $\kappa\lambda$ is less than its Tangent $\kappa\phi$; the Arch $\kappa\mu$ therefore is less than $\kappa\kappa$. But from this PROP. $\kappa\lambda$ is less than $\kappa\mu$, and consequently less than the right Line $\kappa\kappa$. Through the Point o draw the right Line $o\omega$, as per Scheme, then κo is less than $\kappa\omega$, and so $\kappa\lambda$ is less than $\kappa\omega$. In like manner $\kappa\lambda$ is less than κv ; and so we may proceed infinitely, the Arch $\kappa\lambda$ approaching nearer and nearer to an Equality with some part of the Line $\kappa\phi$, in proportion as the Distances κa , κc , κd , &c. increase, that is, in proportion as the Angle of Contact $\phi\kappa\lambda$ is diminished. But it is to be observed, that these Approximations infinitely continued will never make one Tangent Arch equal to another, or to a Part of a Tangent right Line: For if a Tangent Line $\kappa\lambda$ be drawn to the Arch $\kappa\lambda$, and from the Center a an Arch $\lambda\epsilon$ is described, it is sufficiently evident, that a Line drawn from the Center a , of the Arch $\kappa\lambda$ shall touch the Arch $\lambda\epsilon$ in λ , and a Line drawn from the Center

Center of the Arch $x\mu$ to the Point μ , shall likewise be a Tangent to the Arch λs . But the Arch $x\lambda$ is still less than the Arch $x\mu$, as may be inferred from PROP. I. and so by this Method alone we can never come nearer to an Equality between the Arch $x\lambda$, and any superior Tangent Arch, or Tangent right Line, than the x Arch, or Tangent Line terminated by the Arch λs .

L E M M A.

Fig. III.

IF the Arch $\beta\lambda$ is supposed equal to the Tangent Arch βp , and to the Tangent right Line βq , then a Circle may be described through the Points λ, p, q , concave towards the Line $\beta\mu$, at right Angles to the Line βq ; and if concentric Arches are described from the Point μ , (the Center of the Arch $\beta\lambda$) as far as to the Point x , which terminates the Quadrant xz , they shall continually exceed each other.

The first Part of the Lemma is sufficiently evident from Calculation, and may be easily proved to extend itself to any two Tangent Arches and a Tangent right Line; and the second Part is likewise

D

evident

evident from this, that the same Arch of a Circle has every where the same Curvature, and consequently, supposing the *Tangent Arch* βp , equal to $\beta \lambda$, to be drawn infinitely near $\beta \lambda$, then a concentric Arch described from μ , and passing through p , shall be evidently greater than $\beta \lambda$, and so if beyond the Point p , we suppose an Arch $p t$ taken, equal to the infinitely small Arch $p \lambda$, the Inclination or Curvature of the Arch $t p$ shall be the same with that of the Arch $p \lambda$; and therefore I conclude the concentric Arch passing through p , shall be less than the concentric Arch passing through t , &c.

PROPOSITION III. *Fig. 3.* N° I.

Things remaining as in the Lemma, all the Tangent Arches described beyond the Arch βp , and terminated by the Arch $\lambda \phi$, shall be respectively equal to each other, and to the Tangent Line $\beta \phi$.

* From the Point z , which determines the Quadrant $r z$, suppose an Arch to be described concentric to the Arch $\beta \lambda$, and with the same Radius let a Tangent Arch $\beta \epsilon$, from the Point β be described,

* It is necessary to observe, that if the Arch $\gamma \phi$ is greater than a Quadrant, it shall be greater by an Arch less than One Half of the Arch $\lambda \phi$, as is evident; and therefore it will universally follow that the Arch $\beta \lambda$ shall not be greater than a Tangent Arch drawn infinitely near βp , which is all that is requisite to the Demonstration.

scribed, then (from PROP. I. and preceding LEMMA) $\beta \lambda$ is not greater than $\beta \epsilon$, and after the same Manner it may be shewn $\beta \lambda$ is not greater than any Tangent Arch falling within $\beta \lambda, \beta \epsilon$. It is therefore either equal to or less than the Arch $\beta \sigma$. Let then $\beta \lambda$ be less than $\beta \sigma$, and make $\beta \eta = \beta \lambda$, then through the three Points λ, η, ϕ , (from preceding LEMMA :) Let a Circle λ, η, ϕ , be described, this shall fall within the Arch $\lambda \phi$, and consequently shall make $\beta \gamma$, less than $\beta \rho$. But if the Arch $\beta \eta = \beta \lambda$, then from the Principles already established, $\beta \lambda$ shall not be greater than $\beta \gamma$, which is manifestly absurd. Every Tangent Arch therefore, within the Arch $\beta \epsilon$ is equal to the Arch $\beta \lambda$, and after the same Manner we prove every Tangent Arch beyond the Arch $\beta \epsilon$, is equal to the Arch $\beta \lambda$, or the Tangent Line $\beta \phi$. For suppose an Arch $\beta \nu$, described infinitely near the Arch $\beta \epsilon$, then $\beta \epsilon$ shall not be greater than $\beta \nu$, and if it is supposed less, the same Absurdity will ensue, as we have noted in the former Part of this Proposition; and so we may proceed infinitely. All the Tangent Arches therefore beyond the Arch $\beta \lambda$, terminated by the Arch $\lambda \phi$, are equal to each other, and to the right Line, $\beta \phi$. Q. E. D.

C O R O L L A R Y.

Hence if any three Tangent Arches (each not exceeding a Quadrant) are assumed equal to each other, a Circle described through them

them, shall cut off a Part of the common tangent right Line equal to each, and to any Tangent Arch that can be possibly drawn; for if this is denied, one Circle shall cut another in three Points, which is absurd.

Fig. III. No. 2.

Another Method of demonstrating the preceding Proposition.—
 Let a Part $a d$ of the Arch χa be assumed infinitely small, and let a Line be drawn from a to ϕ , the Center of the Arch χm . Then because every Point of the Line ϕm beyond m falls without the Arch χm , the right Line ϕa shall intersect the Arch χm in one Point only, *viz.* in the Point o . Through the Point o describe an Arch touching the Line $\chi \epsilon$ in χ , then the Arch χo shall not be greater than the Arch χd (the Arch χo being less than the Arch χa , as is easily inferred from PROP. I.) for if it is greater by an infinitely small Quantity, it shall be equal to the Arch χa , which is manifestly absurd. In like manner we may proceed with the rest, and demonstrate that every inferior tangent Arch is not greater than it's superior terminated by the same Arch χm , till at length the Arch χm is proved not greater than some Tangent Arch drawn infinitely near it; suppose the Arch χt ; for the Arch χn is less than χt , and if it were possible that by proceeding in
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this manner, the infinitely small Part of some Arch falling without the Arch ρm , should never be in the right Line ρm ; an infinitely small Part (suppose $m n$) must at length remain, or by continuing the Process, a Part still less than that, and so on, *ad infinitum*. χm , therefore, is not greater than χt , and χt is not greater by the above Method than χd ; therefore χt and χm are equal, and so of the rest.

The Two preceding Methods of Demonstration might have been omitted on Account of the superior Accuracy and Elegance of the following; however as they serve to illustrate the Method of Reasoning adopted in this Work, they are on that account retained.

Fig. III. No. 1.

If the Arch ρo is supposed equal to the Arch ρt , equal to the right Line ρq , then any other Arch ρs , falling within ρo , ρt shall be equal to the right Line ρq .

Let a Line be drawn from the Point o , meeting the Arch ρs produced in a , and suppose the Arch $s a$ to be divided by a continual Bisection into an infinite Number of equal Parts, and let $a b$ be one of those Parts; from the Point b , let the Line $b \mu$ be drawn,

E

this

this shall intersect the Arch $\alpha \rho$ in some Point i : Through the Point i let the tangent Arch βi be described, then the Arch βo shall not be greater than the Arch βi , for if it is greater by a Quantity equal to the Arch $i \rho$, it shall be equal to the Arch βy , that is (because ab is infinitely small, and iy is less than ab , as is easy to infer from PROP. I.) if it is greater by a Quantity less than one infinitely small, an Absurdity shall ensue; in like manner the Arch βi is proved not greater than βt , by supposing bd , in the Arch sa , equal to ab , and so we may proceed infinitely, till at length the Arch βo is demonstrated not greater than the Arch βs . In like Manner the Arch βs is not greater than βe , and so must be equal to it and to the right Line $\beta \rho$. Q. E. D.

PROPOSITION IV. *Fig. 4.*

TO Cut any given Angle $s \rho R$, not exceeding a right Angle, in any given Ratio.

Let it be required to be divided in the Ratio of the Line ρQ to the Line ρs . Describe from the Center ρ , the Arches $Q \rho$, $s R$, and

and bisect the Line Q_0 , in the Point λ , and from the same Centre ϕ , describe the Arch $\lambda x a$. At the Point λ , with a Radius equal to ϕQ_0 , describe the Tangent Arch as in the Figure, and at the same Point with a Radius double to ϕQ_0 , another Tangent Arch, then $\frac{1}{2}$ of the Tangent Arch λx shall be equal to $\frac{1}{2}$ of the Second, and $\frac{1}{3}$ of the Third, as is evident. Through the Points of Equality, describe a Circle; this shall make $\frac{1}{2}$ of the Arch Q_0 equal to some Part of the Tangent Arch described with the Radius ϕs , and so a Part of the Arch $s R$ may be taken equal to the Arch Q_0 , which was required to be done.

COROLLARY.

Hence, by a double Operation, any obtuse Angle may be cut in a given Ratio; and hence, by the Way, it is easy to find a right Line equal to a given circular Arch.



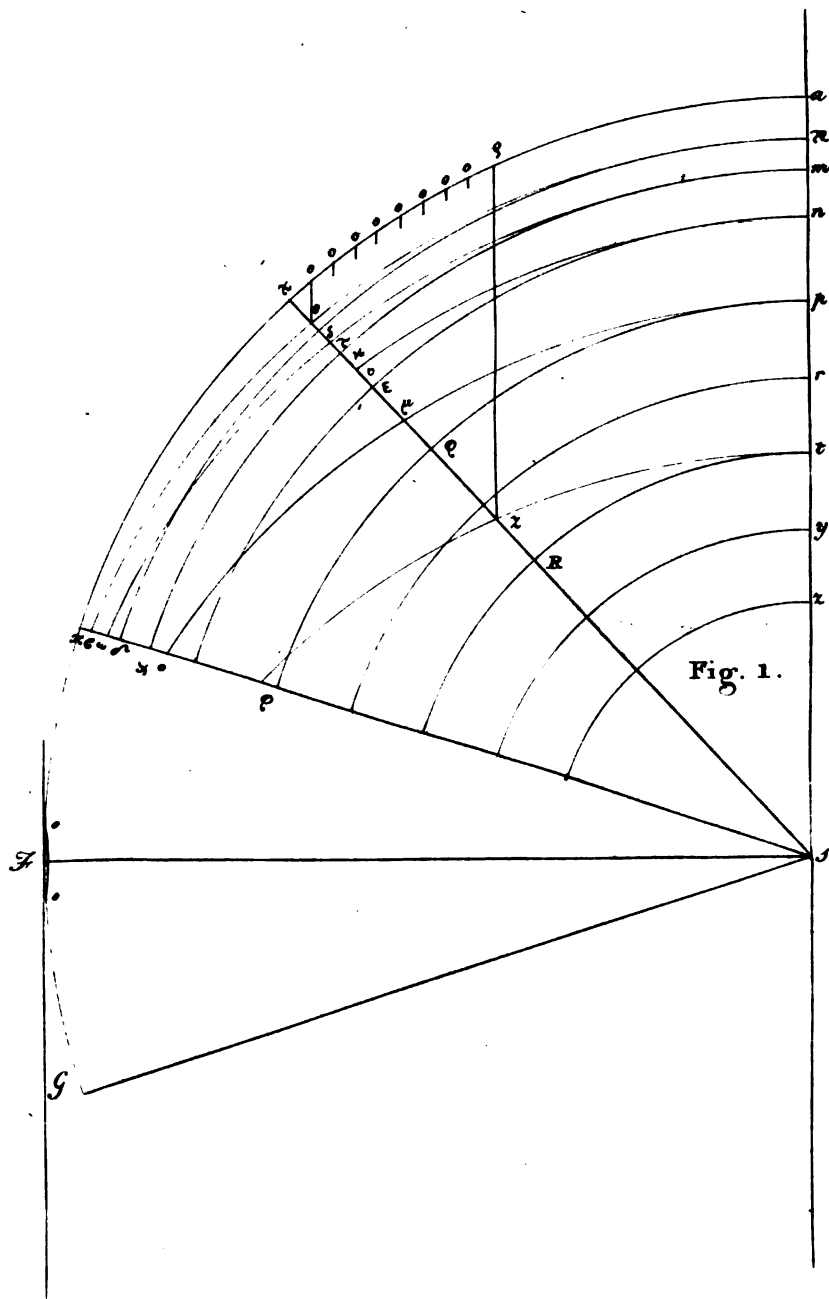
SCHOLIUM.

THE Quadratrix and Spiral, it is well known, have been severally applied, by very able Mathematicians, to a Solution of the above Proposition; however, not with sufficient Success to stop all future Attempts; the Construction of each depending on the
continual

continual Bisection of an Arch, which although very simple in Theory, yet when continued beyond a certain Degree of Minuteness is obnoxious to the greatest Errors in Practice; not to mention that after all, the Rectification of the Circle cannot be geometrically obtained by these Inventions, and that in the Opinion of the best Judges they are mechanical Curves. Perhaps when the Errors arising from such a Construction are fairly estimated, the preceding Method will appear no contemptible Substitute for this Purpose; since if the Principles on which it is founded are once admitted, the Solution will be natural and easy, and at the same time consistent with the greatest Rigor of Mathematical Demonstration.

It likewise seems probable, that the Rectification of other Curves may be obtained by a similar Method of Reasoning, as the Lemma at the beginning of this Tract extends itself to all kind of Quantities whatever:—But a Critical Examination of this Point would not only exceed the Limits, but likewise be foreign from the original Design of this Work.

T H E E N D.



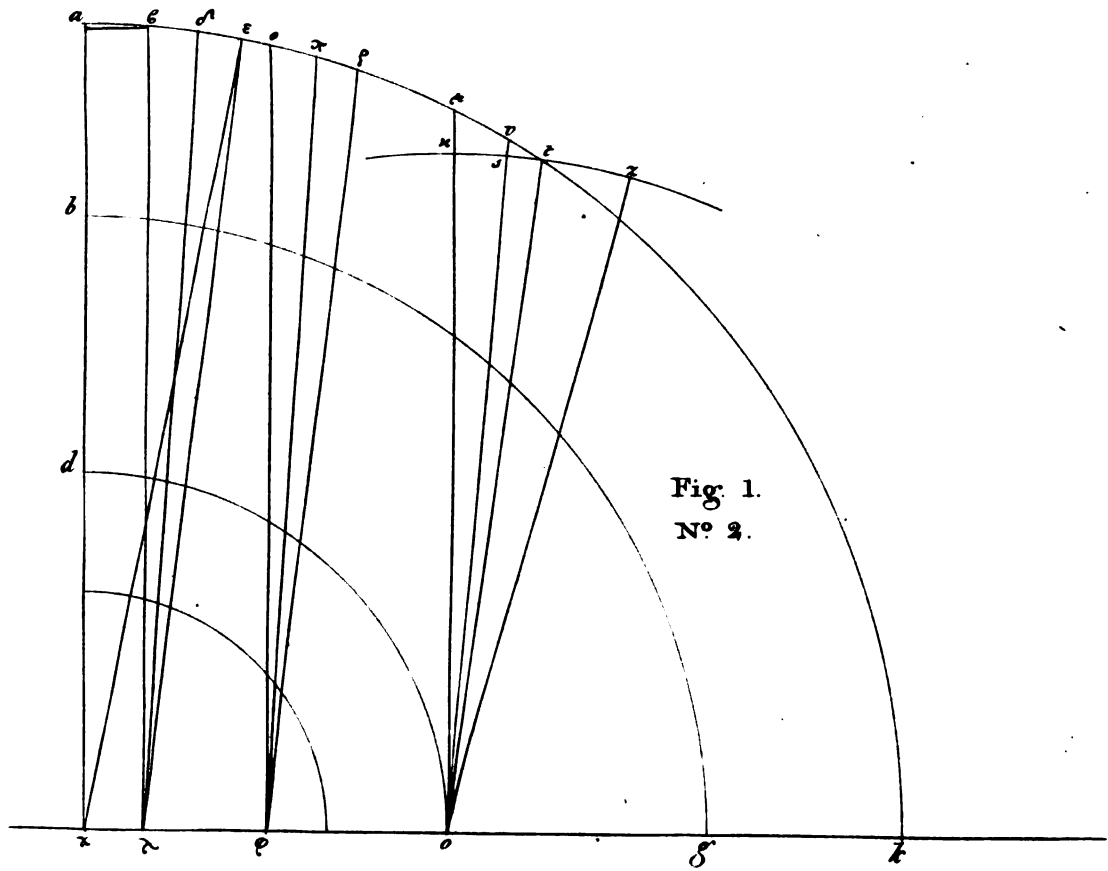
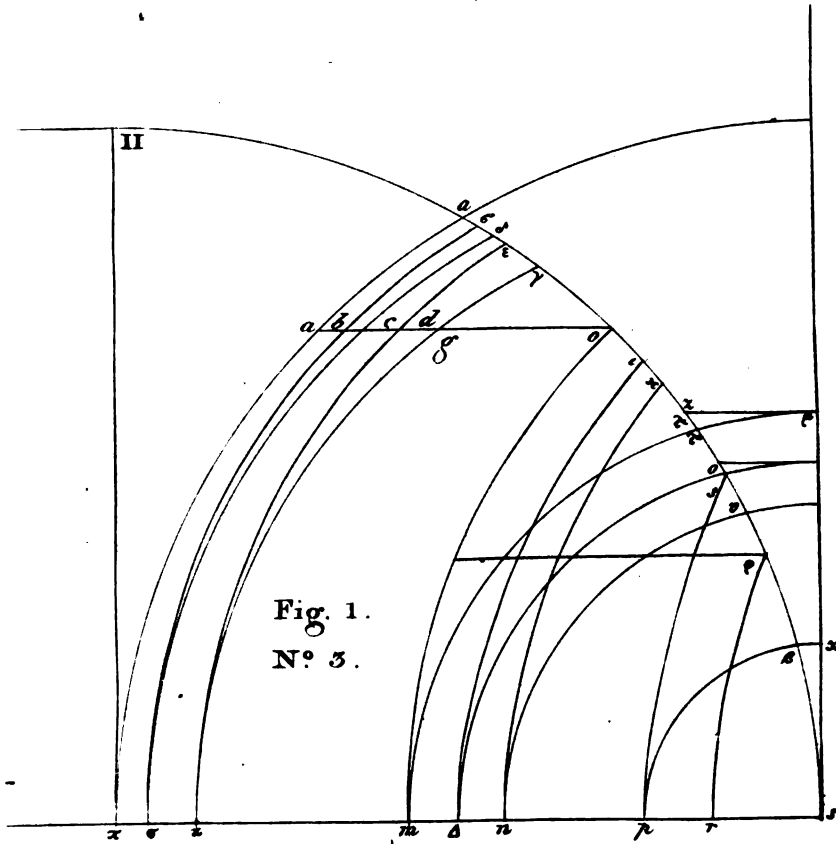


Fig. 1.
N° 2.



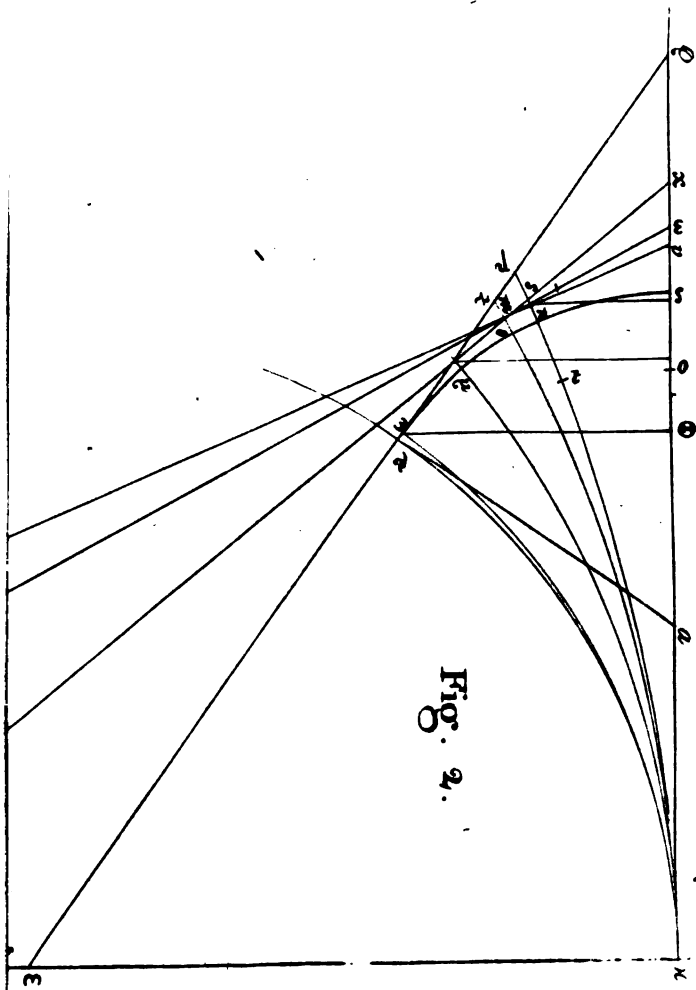


FIG. 2.

